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Current Noise and Long Time Tails in Biased Disordered Random Walks

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We calculate the mean velocity and the velocity correlation function for a random walk with a uniform bias on a disordered chain. We find a new long time tail in the velocity correlation function due to the combined effects of the bias and of the disorder in the site variables. This long time tail persists to a low-frequency cutoff inversely proportional to the square of the bias. By associating the velocity correlation function with the spectrum of current fluctuations, we calculate the excess low-frequency current noise associated with this long time tail. The spectrum of current fluctuations goes as $(I^2/N)f^{-1/2}$, where I is the DC current, N is the number of charge carriers, and f is the frequency. The possible connection to $1/f$ noise is discussed. The calculation is done by a perturbation expansion in the strength of the disorder, but is shown to be exact to all orders for weak enough bias.

KEY WORDS: Random walk; disorder; current fluctuations; long time tails; $1/f$ noise.

1. INTRODUCTION

One-dimensional random walks with static disorder are the subject of considerable current interest.[1-14] The effect of disorder for transition rates

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obeying detailed balance is now well understood.[1–11] These random walks can be used to model the equilibrium properties of one-dimensional disordered conductors.[7,8] By using the Einstein relation, the weak field conductivity of these systems can also be analyzed. On the other hand nonequilibrium properties, such as the spectrum of excess current fluctuations in the presence of a DC current, cannot easily be determined from an equilibrium model. Thus it is important to understand the properties of nonequilibrium random walk models.

In this paper we study a biased disordered random walk governed by the master equation,

$$\frac{dP_n(t)}{dt} = \sum_m T_{nm} P_m(t) \tag{1.1}$$

with the transition matrix T_{nm} given by

$$\begin{aligned} T_{nm} = {} & e^{\epsilon}(W_m/C_m)\delta_{n,m+1} + e^{-\epsilon}(W_{m-1}/C_m)\delta_{n,m-1} \\ & - e^{\epsilon}(W_m/C_m)\delta_{n,m} - e^{-\epsilon}(W_{m-1}/C_m)\delta_{n,m} \end{aligned} \tag{1.2}$$

$P_n(t)$ is the probability that the walker is at the nth lattice site at time t. W_n is the transition rate between sites n and $n+1$, and C_n is the well depth at the nth site. The equilibrium concentration of walkers at site n is proportional to C_n. The factor ϵ determines the bias. For a conductivity problem it can be identified with the ratio of the electrical energy gained per hop to the thermal energy

$$\epsilon = e\mathscr{E}l/2kT \tag{1.3}$$

where e is the charge per carrier, $\mathscr{E}$ is the electric field, l is the lattice spacing, k is Boltzmann's constant, and T is the temperature. The choice for ϵ given in (1.3) insures that P_n would take the correct local equilibrium form, proportional to $C_n \exp(nel\mathscr{E}/kT)$, if the chain were closed at both ends.

The random walk described by (1.1) and (1.2) is disordered if the parameters W_n and C_n are chosen to vary randomly with n. If C is constant and W is random we have bond disorder. If W is constant and C is random we have site disorder. In the present paper we consider primarily the case of site disorder. We study the site disorder problem first because it is mathematically simpler, and more clearly displays the new, specifically nonequilibrium effects of disorder. Results for bond disorder will be presented in a subsequent paper.

The equilibrium ($\epsilon = 0$) properties of (1.1) and (1.2) for the case of bond disorder have been studied by a number of authors.[1,2,4–10] One of the interesting results in this case is power law behavior in the frequency-dependent conductivity or, equivalently, in the velocity correlation function. This "long time tail" effect has been found in several other stochastic models,[15,16] and is closely related to the long time tails found in fluids[17] and Lorentz gases.[18]

Derrida and Orbach[12] have studied the nonequilibrium properties of (1.1) and (1.2) for the case of bond disorder. They calculated the frequency-dependent conductivity in the presence of a steady current. An interesting feature of their result is that it separates into two frequency regimes. At the lowest frequencies the motion of the particles is dominated by a steady drift, whereas for intermediate frequencies it is dominated by diffusion. In the diffusion-dominated regime there are long time tail effects similar to those found in equilibrium, whereas in the drift-dominated regime these are replaced by a behavior which is analytic in frequency.

The equilibrium properties of (1.1) and (1.2) for the case of site disorder have been studied in Refs. 3 and 9 and have certain simplifying features. Because of the spatial isotropy of the master equation[16] there are no long time tails in the velocity correlation function or the frequency-dependent conductivity. The effect of site disorder first appears in the fourth cumulant (3), $\langle x^4(t)\rangle - 3\langle x^2(t)\rangle^2$ or, equivalently, in a frequency- and wave-number-dependent diffusion coefficient. By adding a bias, spatial isotropy is broken and, as we shall calculate, long time tails appear in the velocity correlation function and in the current fluctuations. As in the case of bond disorder, these long time tails arise only in the diffusion dominated regime.

An intriguing aspect of this new long time tail is that it leads to current fluctuations which are similar in several respects to the ubiquitous phenomenon of $1/f$ noise.[20] In both cases the strength of fluctuations is proportional to the square of the current, and inversely proportional to the number of carriers. In both cases the spectrum is a power law over a number of decades in frequency, but here the power law is $f^{-1/2}$ rather than f^{-1}. In the last section we discuss the connection between this result and a recent model[21,22] for $1/f$ noise based on a random walk with a random bias.[14]

The outline of the paper is as follows: In Section 2 the response function is defined and related to the moments of the displacement, the velocity correlation function, the current, and the current fluctuations. In Section 3 a formally exact expression for the response function in terms of the fluctuating quantities is presented. In Section 4 and Appendix A results are presented for the quantities of physical interest calculated to second order in the fluctuating quantity $(C_m - 1)$. In Section 5 we show that these results hold to all orders in perturbation theory for sufficiently weak bias. The paper closes with a discussion.

2. FORMALISM

In this section we define the response function, and show how it can be used to calculate the various quantities of physical interest; the moments of the displacement, the mean velocity, the velocity correlation function, the steady state current, and current fluctuations. We begin by formally solving the master equation using Laplace transforms. Let

$$\tilde{P}_n(z) = \int_0^\infty e^{-zt} P_n(t)\, dt \tag{2.1}$$

The formal solution to the master equation is

$$\tilde{P}_n(z) = \sum_m G_{nm}(z) P_m(0) \tag{2.2}$$

where $G_{nm}(z)$ is the Green's function for the walk. Using matrix notation $\mathsf{G} = G_{nm}$ is defined by

$$\mathsf{G}(z) = [z\mathbf{1} - \mathsf{T}]^{-1} \tag{2.3}$$

where $\mathbf{1}$ is the identity matrix.

Suppose that Q_m is the stationary solution to the master equation corresponding to a steady current. We choose the normalization such that

$$\sum_{m=1}^{M} Q_m = 1$$

where M is the number of sites on the chain. The response function $f(q, z)$ is defined as the product of the Green's function and the stationary distribution,

$$f(q, z) = \sum_{m\,n} e^{iql(n-m)} F_{nm}(z) \tag{2.4}$$

with

$$F_{nm}(z) = \langle G_{nm}(z)\, Q_m \rangle \tag{2.5}$$

The brackets indicate an average over the probability distribution for the W's and the C's.

The mean displacement $\mu_1(t)$ and the mean square displacement $\mu_2(t)$ can be obtained from the response function by differentiation with respect to the wave number q. Their Laplace transforms are given by

$$\tilde{\mu}_1(z) = \sum_{m\,n} l(n-m) F_{nm}(z) = -i \frac{\partial}{\partial q} f(q, z)\Big|_{q=0} \tag{2.6}$$

and

$$\tilde{\mu}_2(z) = \sum_{m\,n} l^2(n-m)^2 F_{nm}(z) = -\frac{\partial^2}{\partial q^2} f(q, z)\Big|_{q=0} \tag{2.7}$$

In these expressions we have averaged first over the stationary distribution and then over the disorder. The average over the disorder can either be interpreted physically as representing the results of an experiment performed on a large number of parallel chains, or this average can be viewed simply as a mathematical convenience. In general we expect that averaging over a single long chain should be equivalent to averaging over an ensemble of shorter chains.

With an additional assumption, $\mu_1(t)$ and $\mu_2(t)$ can also be used to calculate the current fluctuations. We assume that the instantaneous current can be identified with the spatially averaged velocities of the charge carriers,

$$i(t) = (e/L) \sum_{j=1}^{N} v_j(t) \tag{2.8}$$

where $v_j(t)$ is the velocity of the jth carrier, L is the length of the sample, and N is the total number of charge carriers. Since the carriers move independently the mean current, I, can be related to $\mu_1(t)$,

$$I = (eN/L)\langle v \rangle \equiv (eN/L)\, d\mu_1(t)/dt \tag{2.9}$$

We can also relate the spectral density of the current fluctuations to $\mu_2(t)$. The spectral density of current fluctuations, $P_I(\omega)$ is defined by

$$P_I(\omega) = 4\int_0^\infty \left[\langle i(t)i(0)\rangle - I^2\right]\cos(\omega t)\, dt \tag{2.10}$$

For independent charge carriers $P_I(\omega)$ is proportional to the velocity correlation function $\phi(t)$ defined by

$$\phi(t) = \langle v(t)v(0)\rangle - \langle v\rangle^2 = \frac{1}{2}\, \frac{d^2}{dt^2}\, \mu_2(t) - \left[\frac{d\mu_1(t)}{dt}\right]^2 \tag{2.11}$$

Combining Eqs. (2.8)–(2.11) we obtain

$$P_I(\omega) = (4e^2N/L^2)\mathrm{Re}\,\tilde{\phi}(i\omega) \tag{2.12}$$

3. RESPONSE FUNCTION

In this section we specialize to the case of site disorder, and derive an expression for the response function in terms of the moments of the random variables, $\{C_n\}$. We assume the distribution of the C_n to be sufficiently nonsingular that all moments exist. Without loss of generality we choose $\langle C_n\rangle \equiv 1$, and $W_n \equiv \nu$. We assume that the C_n are independently and identically distributed random variables. In order to find a moment expansion for the response function, we split the transition matrix T and the Green's function G into a bare part and a fluctuating part. Let

$$\Delta\mathsf{T} = \mathsf{T} - \mathsf{T}^0 \tag{3.1}$$

where T^0 is the transition matrix for a uniform system with $C_n \equiv 1$ and $W_n \equiv \nu$ [see Eq. (1.2)],

$$T^0_{nm} = \nu\left[e^{\epsilon}\delta_{n,m+1} + e^{-\epsilon}\delta_{n,m-1} - 2\cosh\epsilon\delta_{n,m}\right] \tag{3.2}$$

From (2.3) and (3.1) it is straightforward to verify that the Green's function can be written

$$\mathsf{G} = \mathsf{G}^0 + \mathsf{G}^0\Delta\mathsf{T}\left[1 - \mathsf{G}^0\Delta\mathsf{T}\right]^{-1}\mathsf{G}^0 \tag{3.3}$$

where G^0 is the bare Green's function,

$$\mathsf{G}^0 = \left[z1 - \mathsf{T}^0\right]^{-1} \tag{3.4}$$

For the site disorder problem $\Delta\mathsf{T}$ takes the simple form,

$$\Delta\mathsf{T} = \mathsf{T}^0\Delta\mathsf{S} \tag{3.5}$$

where the matrix $\Delta\mathsf{S}$ is given by

$$\Delta S_{nm} = -\delta_{nm}\left[(C_m - 1)/C_m\right] \tag{3.6}$$

Thus (3.3) can be rewritten

$$\mathsf{G} = \mathsf{G}^0 + \mathsf{G}^0\mathsf{T}^0\Delta\mathsf{S}\left[1 - \mathsf{G}^0\mathsf{T}^0\Delta\mathsf{S}\right]^{-1}\mathsf{G}^0. \tag{3.7}$$

It is very useful to write the Green's function explicitly in terms of the basic fluctuating quantities $\{C_n - 1\}$. Defining the matrix $\delta\mathsf{C}$ by

$$\delta C_{nm} = \delta_{nm}(C_m - 1) \tag{3.8}$$

and using the identity

$$\mathsf{G}^0\mathsf{T}^0 = z\mathsf{G}^0 - 1 \tag{3.9}$$

we can write the fluctuating part of the Green's function as

$$\Delta\mathsf{G} \equiv \mathsf{G} - \mathsf{G}^0 = -(z\mathsf{G}^0 - 1)\delta\mathsf{C}\left[1 + z\mathsf{G}^0\delta\mathsf{C}\right]^{-1}\mathsf{G}^0 \tag{3.10}$$

The form of the Green's function in (3.10) will be exploited in Section 5 to show that results presented in Section 4 are exact for small ϵ even when δC has large fluctuations. The method of writing the Green's function in terms of the basic fluctuating quantities was first used by Zwanzig[5] in analyzing the unbiased bond disorder problem, and later by Denteneer and Ernst[9] for both the unbiased bond disorder and the unbiased site disorder problem.

In order to calculate the response function we need to know both the Green's function and the steady state solution, Q_n, of the master equation. For the site disorder problem this solution can be found by direct substitution, and is given by

$$Q_m = C_m/M \tag{3.11}$$

where M is the total number of lattice sites and we have normalized the sum over Q_m to unity. Substituting (3.11) into (2.5) we obtain

$$M\mathsf{F} = \mathsf{G}^0 + \langle \Delta \mathsf{G} \rangle + \langle \Delta \mathsf{G} \delta \mathsf{C} \rangle \tag{3.12}$$

where $\Delta\mathsf{G}$ is given by (3.10). After some matrix algebra this can be written

$$M\mathsf{F} = \mathsf{G}^0 + (z\mathsf{G}^0 - 1)\mathsf{I}(z\mathsf{G}^0 - 1) \tag{3.13}$$

where the matrix I is defined by

$$\mathsf{I} = \langle \delta \mathsf{C}\left[1 + z\mathsf{G}^0 \delta \mathsf{C}\right]^{-1} \mathsf{G}^0 \delta \mathsf{C} \rangle \tag{3.14}$$

In (3.12)–(3.14) we have used $\langle \delta C \rangle = 0$. Since the average is translationally invariant we can explicitly take the Fourier transform and obtain the response function $f(q,z)$ defined in (2.4),

$$f(q,z) = g(q,z) + \left\{ g(q,z)\left[z - g^{-1}(q,z) \right] \right\}^2 I(q,z) \tag{3.15}$$

In (3.15), $g(q,z)$ is the Fourier transform of the bare Green's function

$$\begin{aligned} g(q,z) &= \sum_n e^{iqln} G^0_{n0}(z) \\ &= \left\{ z - 2\nu\left[\cosh(iql + \epsilon) - \cosh \epsilon \right] \right\}^{-1} \end{aligned} \tag{3.16}$$

and $I(q,z)$ is the Fourier transform of the right-hand side of (3.14).

We can now use (2.6)–(2.7) to calculate the mean displacement and mean squared displacement. Since the leading term in an expansion of $(z - g^{-1})$ in powers of q is of order q, the first moment of the displacement is independent of the disorder, and is given by

$$\tilde{\mu}_1(z) = (2\nu l \sinh \epsilon)/z^2 \equiv \langle v \rangle / z^2 \tag{3.17}$$

For the same reason, the mean square displacement depends only on $I(0,z)$ and can be written

$$\tilde{\mu}_2(z) = 2D(\epsilon)z^{-2} + 2\langle v \rangle^2 z^{-3} + 2\langle v \rangle^2 I(0,z) z^{-2} \tag{3.18}$$

where $D(\epsilon) = \nu l^2 \cosh \epsilon$ is the diffusion coefficient. The first two terms in (3.18) are expected in the absence of disorder due to diffusion and drift, respectively. The third term gives the interesting extra contribution when both a bias and disorder are present.

4. RESULTS

In this section we present results for the physically important quantities; the current, the velocity correlation function, and the current fluctuations. The results presented here are valid only to second order in δC. In Section 5 we show that these results are exact to all orders in δC in the Ohmic regime where ϵ is small.

Using (2.9) and taking the inverse transform of (3.17) we find that the steady current is given by

$$I = Ne\langle v\rangle / L \tag{4.1}$$

with $\langle v\rangle$ given by (3.17). In general ϵ will be small since it measures the ratio of electric energy gained per hop to thermal energy. If (3.17) is expanded to first order in ϵ we recover Ohm's law. Equation (4.1) is conveniently written in the form

$$I = ne^2\mu A\mathscr{E} = V/R \tag{4.2}$$

where the mobility μ is given by

$$\mu = \nu l^2/kT \tag{4.3}$$

A is the cross-sectional area of the sample, n is the density of carriers, V is the voltage, and

$$R = (L/\mu ne^2 A) \tag{4.4}$$

is the resistance of the sample. Note that (4.3) is the Einstein relation since $\nu l^2 = D_0$ is the equilibrium diffusion coefficient.

To calculate the velocity correlation function we must expand $I(0, z)$ to order δC^2. Carrying out this expansion on the Fourier transform of (3.14) we obtain

$$I(0, z) = \Delta_c G^0_{nn}(z) \tag{4.5}$$

where

$$\Delta_c = \langle (C_n - 1)^2\rangle \tag{4.6}$$

In deriving (4.5) we have used the fact that the C_n's are not correlated from site to site. In Appendix A we derive a general expression for the Green's function which gives

$$G^0_{nn}(z) \equiv \psi(z) = \left[z^2 + 4z\nu\cosh\epsilon + 4\nu^2\sinh^2\epsilon\right]^{-1/2} \tag{4.7}$$

Using (2.11), (3.17)–(3.18), and (4.5)–(4.7) the velocity correlation function is given by

$$\tilde{\phi}(z) = D(\epsilon) + \langle v\rangle^2 \Delta_c \psi(z) \tag{4.8}$$

where

$$D(\epsilon) = \nu l^2 \cosh \epsilon = D_0 \cosh \epsilon \tag{4.9}$$

Notice that $\tilde{\phi}(z)$ is independent of z when $\langle v \rangle = 0$. This is due to the fact that, in the absence of a bias, the random walker always jumps with probability 1/2 to the right or to the left so that the velocity of the walker is delta correlated.[16] This symmetry makes the random site problem considerably easier to solve than the random bond problem.

In the presence of a drift there are several frequency ranges of interest. We consider only low frequencies $|z| \ll \nu$, so that we average over many hops. For the disordered random walk with drift this low-frequency range is further subdivided depending on whether drift or diffusive spreading dominates the sampling of the randomness. The crossover frequency is given by

$$\omega_c = \epsilon^2 \nu \ll \nu \tag{4.10}$$

Introducing a dimensionless frequency variable

$$x = z/\omega_c \tag{4.11}$$

and expanding $\tilde{\phi}(z)$ to first order in ϵ holding x fixed yields

$$\tilde{\phi}(z) = D_0\left[1 + 2|\epsilon|\Delta_c(x+1)^{-1/2}\right] \tag{4.12}$$

In the drift-dominated regime where $|x| \ll 1$, (4.12) reduces to

$$\tilde{\phi}(z) = D_0[1 + 2|\epsilon|\Delta_c] \tag{4.13}$$

We can interpret (4.13) as a nonequilibrium renormalization of the diffusion coefficient. An interesting feature of (4.13) is that it is nonanalytic in the bias strength at $\epsilon = 0$.

In the diffusion-dominated regime where $\omega_c \ll |z| \ll \nu$, (4.12) reduces to

$$\tilde{\phi}(z) = D_0\left[1 + 2\epsilon^2\Delta_c(\nu/z)^{1/2}\right] \tag{4.14}$$

The $z^{-1/2}$ behavior in the diffusion-dominated regime is a new nonequilibrium long time tail phenomenon.

The physical interpretation of these results is best appreciated by looking at the spectrum of current fluctuations. For simplicity we look only at the drift- and diffusion-dominated regimes of (4.13) and (4.14). To relate $\tilde{\phi}(z)$ to the current fluctuations we use (2.12), (4.3), (4.4), and (4.9). In the drift-dominated regime $\omega \ll \omega_c$ we obtain from (4.13)

$$P_I(\omega) = 4kTR^{-1}[1 + 2|\epsilon|\Delta_c] \tag{4.15}$$

which is a renormalization of the equilibrium Johnson noise.

In the diffusion dominated regime $\omega_c \ll \omega \ll \nu$, we obtain from (4.14)

$$P_I(\omega) = 4kTR^{-1}\left[1 + \sqrt{2}\,\epsilon^2\Delta_c(\nu/\omega)^{1/2}\right] \tag{4.16}$$

which contains an interesting frequency dependence. In deriving (4.14) and (4.16) we expanded to first order in ϵ for x fixed, but the result is also correct to second order in ϵ for frequencies in the diffusion-dominated regime. To see the physical content of (4.16) more clearly we rewrite it using (3.17), (4.1), and (4.9) to give

$$P_I(\omega) = 4kTR^{-1} + \sqrt{2}\,\Delta_c(I^2/N)(\omega\nu)^{-1/2} \tag{4.17}$$

The range of frequencies over which this power law behavior can be exhibited in $P_I(\omega)$ can be quite large. The hopping frequency ν is typically rather high. The crossover frequency ω_c can be calculated in the same way we obtained (4.17) from (4.16), and is given by

$$\omega_c = I^2R/4NkT \tag{4.18}$$

We see that ω_c is just the ratio of the power dissipated per carrier to the thermal energy. At low currents in the Ohmic regime this will be a very low frequency so that the $\omega^{-1/2}$ behavior of the current fluctuations may be present over several decades of frequency. Our simple model system exhibits "$1/\sqrt{f}$ noise."

5. EXACT RESULTS IN THE OHMIC REGIME

In Section 3 we derived a general expression for $\mu_2(z)$ in terms of the quantity $I(0,z)$ which depended on all the moments of the random matrix $\delta\mathrm{C}$. In Section 4 we expanded $I(0,z)$ up to second order in $(C_n - 1)$ and then further expanded the results to first order in ϵ while holding $x = z/\nu\epsilon^2$ fixed. In this section we examine the contribution of higher powers of $\delta\mathrm{C}$.

From (2.11) and (3.18) we obtain the full expression for $\tilde{\phi}(z)$,

$$\tilde{\phi}(z) = \nu l^2\cosh\epsilon + 4\nu^2 l^2 I(0,z)\sinh^2\epsilon \tag{5.1}$$

Expanding (3.14) as a (matrix) geometric series yields a moment expansion for $I(0,z)$

$$I(0,z) = \sum_{r=0}^{\infty}(-z)^r\left\langle \sum_{n=-\infty}^{\infty}\left[(\delta\mathrm{C}\mathrm{G}^0)^{r+1}\delta\mathrm{C}\right]_{n0}\right\rangle \tag{5.2}$$

To illustrate how the terms in this series contribute to $\tilde{\phi}(z)$, consider the coefficient $I^{(2)}$, of z^2 in (5.2),

$$I^{(2)} = \sum_{n=-\infty}^{\infty}\langle[\delta\mathrm{C}\mathrm{G}^0\delta\mathrm{C}\mathrm{G}^0\delta\mathrm{C}\mathrm{G}^0\delta\mathrm{C}]_{n0}\rangle \tag{5.3}$$

Since $\langle C_m - 1 \rangle = 0$ and $\langle (C_m - 1)(C_n - 1) \rangle = \Delta_c \delta_{mn}$ this term can be factorized in exactly four ways.

Using the expression for G^0 obtained in the Appendix,

$$G^0_{n+k,n}(z) = e^{k\epsilon} \chi(z)^{|k|} \psi(z) \tag{5.4}$$

we obtain

$$I^{(2)} = \psi^3 \{ D_4 + \Delta_c^2 [P_{11} + P_{02} + P_{13}] \} \tag{5.5}$$

where P_{jk} represents geometric sums of the χ factors and Boltzmann factors,

$$P_{jk} = \sum_{n=-\infty}^{\infty} e^{jn\epsilon} \chi^{|kn|} = 1 + (e^{-j\epsilon} \chi^{-k} - 1)^{-1} + (e^{j\epsilon} \chi^{-k} - 1)^{-1} \tag{5.6}$$

and

$$D_4 = \langle (C_n - 1)^4 \rangle - 3\Delta_c^2 \tag{5.7}$$

is the fourth cumulant of the fluctuation parameter.

We now expand $I^{(2)}$ to lowest order in ϵ holding x fixed. To this order $\psi(z)$ and $\chi(z)$, given in (A7) and (A8) reduce to

$$\psi(z) = (2\nu|\epsilon|)^{-1} (x+1)^{-1/2} + O(\epsilon^0) \tag{5.8}$$

and

$$\chi(z) = 1 - |\epsilon| (x+1)^{1/2} + O(\epsilon^2) \tag{5.9}$$

Using (5.9) the sums P_{jk}, to leading order in ϵ, are given by

$$P_{jk} = 2|\epsilon|^{-1} k (x+1)^{1/2} [k^2 (x+1) - j^2]^{-1} + O(\epsilon^0) \tag{5.10}$$

Combining (5.5), (5.8), and (5.10) we obtain

$$z^2 I^{(2)} = \frac{\Delta_c^2 x^2}{4\nu(x+1)} \left[\frac{1}{x} + \frac{1}{2x+2} + \frac{3}{9x+8} \right] + O(|\epsilon|) \tag{5.11}$$

By the same reasoning that leads to (5.11) we obtain

$$z I^{(1)} = [D_3 / 4\nu(x+1)] + O(|\epsilon|) \tag{5.12}$$

and

$$I^{(0)} = \psi \Delta_c = (\Delta_c / 2\nu|\epsilon|)(x+1)^{-1/2} + O(\epsilon^0) \tag{5.13}$$

In (5.12) D_3 is the third cumulant of $(C_n - 1)$.

Comparing (5.11), (5.12), and (5.13) we see that only $I^{(0)}$ contributes to $\tilde{\phi}(z)$ to order $|\epsilon|$, whereas $I^{(1)}$ and $I^{(2)}$ both contribute to order ϵ^2. We now show that no higher moments contribute to $\tilde{\phi}(z)$ to order $|\epsilon|$. The general term $I^{(r)}$ is proportional to ψ^{r+1}. In addition there are at least

$q_r + 1$ Kronecker δ functions occurring in the factorization of the average owing to the fact that $\langle \delta \mathbf{C} \rangle = \mathbf{0}$. The quantity q_r denotes the greatest integer less than $(r/2)$. Each δ function removes one summation implicit in the matrix product so that there are, at most, $(r - q_r)$ sums; each of the form P_{jk} with $k > j$. Using (5.8)–(5.10) we can put an upper bound on the ϵ dependence of $z^r I^{(r)}$

$$z^r I^{(r)} \lesssim |\epsilon|^{(q_r - 1)} \tag{5.14}$$

The asymptotic inequality (5.14) proves that the expression for $\tilde{\phi}(z)$ given in (4.12) is exact to order $|\epsilon|$ in an expansion in which both $x = z/\nu\epsilon^2$ and the distribution of the fluctuating parameters are held fixed. Similarly (4.15) and (4.16) are asymptotically correct to order $|\epsilon|$, and (4.16) is correct to order ϵ^2 for frequencies in the diffusion-dominated regime $\omega \gg \nu\epsilon^2$.

If ϵ is small but finite we can estimate when the lowest-order term in the perturbation theory dominates the higher order terms by comparing (5.11) and (5.13). We see that the ratio of the Δ_c to the Δ_c^2 term goes like $\epsilon\Delta_c x^{1/2}$ for $x \gg 1$. Thus the perturbative results of Section 4 are expected to be valid when $\epsilon\Delta_c x^{1/2}$ is small. In this regime the current noise due to the disorder is much smaller than the Johnson noise. However it may be detectable because of its unique frequency dependence. It would be interesting to construct a nonperturbative theory of current noise due to quenched disorder.

6. DISCUSSION

We have calculated a new "long time tail" in the velocity correlation function associated with the combined effects of site disorder and a weak bias. This is related to the long time tail which occurs in equilibrium in the fourth cumulant of the displacement.[3] In both cases the fluctuation correction arises from the diffusive sampling of the randomness. The new long time tail found here differs from equilibrium long time tails in that it appears only above a crossover frequency, ω_c given in (4.18). Below this frequency the behavior of the velocity correlation function is analytic in frequency though nonanalytic in the strength of the bias.

We have calculated the same quantities for the random bond problem. The results are more complicated and yield different power law behavior largely due to the more complicated structure of the nonequilibrium steady state[12] in the combined presence of a bias and bond disorder. These results will be presented in a subsequent paper.[23]

It is interesting to examine the connection between the present disordered random walk model, the phenomenology of "$1/f$ noise," and other recent theoretical attempts[21,22] to relate $1/f$ noise to disordered random walks. In the present model we have made a specific assumption [see (2.8), (2.11), and (2.12)] relating the current noise spectral density to the mean square displacement of a charge carrier. In the diffusion-dominated regime we obtain a proportionality of $P_I(\omega)$ to (I^2/N), in agreement with phenomenology. This emphasizes that excess current noise is observed in the current noise spectral density only in the presence of a DC current. [It is true that $1/f$ noise can be observed at equilibrium,[24] but the equilibrium quantity which exhibits this noise is a complicated four time voltage or current correlation function.[25,26]] On the other hand our calculation gives an $f^{-1/2}$ frequency dependence, which is quite far from the nearly f^{-1} dependence typically observed.

The recent papers by Marinari et al.[21,22] give a calculation on the spectrum of the displacement of a random walker on a chain with a random bias.[14] At each site there is a probability π to jump to the right and $1-\pi$ to jump to the left. The probability π is itself uniformly distributed in the range from zero to one at each site. The resulting spectrum is nearly proportional to $1/f$. In this model the displacement of the random walker cannot be the position of a charge carrier, but must be some more abstract quantity such as the resistance of the sample. It is interesting to search for physical models in which the resistance would perform such a random walk.

To summarize, we have calculated the velocity correlation function and the spectrum of current fluctuations for a site disordered chain with a net current flow. We have obtained a new long time tail associated with the combined effects of disorder and current flow, and the resulting current fluctuations have some features in common with the ubiquitous phenomenon of $1/f$ noise. It remains to be seen if the connection between random walks in disordered media and $1/f$ noise can be made more explicit in terms of physically realistic random walks.

APPENDIX

In this Appendix we derive an expression for the uniform system Green's function in coordinate space, $G^0_{n+k,n}(z)$. Taking the Fourier transform of (3.4) using (3.2) we obtain the Green's function in q space,

$$G^0(q,z) = \{z - 2\nu[\cosh(iql+\epsilon) - \cosh(\epsilon)]\}^{-1} \tag{A1}$$

Using the Fourier inversion formula we find the following integral expression for $G^0_{n+k,n}(z)$,

$$G^0_{n+k,n} = (l/2\pi)\int_0^{2\pi} dq\, e^{-iqkl} g(q,z) \tag{A2}$$

(A2) can be evaluated by making the substitution $s = \exp(iql)$, yielding

$$G^0_{n+k,n} = (i/2\pi)\int_C ds\, s^{-k}\left[s^2 \nu e^{\epsilon} - s(z + 2\nu\cosh(\epsilon)) + \nu e^{-\epsilon}\right]^{-1} \quad \text{(A3)}$$

where the contour C is a positively oriented unit circle centered at $s = 0$. The roots of the quadratic in the square bracket in (A3) are

$$r_{\pm} = (e^{-\epsilon}/2\nu)\left\{z + 2\nu\cosh(\epsilon) \pm \left[z^2 + 4z\nu\cosh(\epsilon) + 4\nu^2\sinh^2(\epsilon)\right]^{1/2}\right\} \quad \text{(A4)}$$

For $\mathrm{Re}(z) > 0$ only r_- lies within C. Noting that (A3) is invariant under the interchanges $\epsilon \to -\epsilon$ and $k \to -k$ and using the residue theorem we obtain,

$$G^0_{n+k,n}(z) = e^{k\epsilon}\chi(z)^{|k|}\psi(z) \quad \text{(A5)}$$

where

$$\psi(z) \equiv G^0_{nn}(z) = \left[z^2 + 4z\nu\cosh(\epsilon) + 4\nu^2\sinh^2(\epsilon)\right]^{-1/2} \quad \text{(A6)}$$

and

$$\chi(z) = \left[z + 2\nu\cosh(\epsilon) - \psi(z)^{-1}\right]/2\nu \quad \text{(A7)}$$

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